

# RELATIVE LODAY CONSTRUCTIONS AND APPLICATIONS TO HIGHER THH-CALCULATIONS

GEMMA HALLIWELL, EVA HÖNING, AYELET LINDENSTRAUSS, BIRGIT RICHTER,  
AND INNA ZAKHAREVICH

**ABSTRACT.** We define a relative version of the Loday construction for a sequence of commutative  $S$ -algebras  $A \rightarrow B \rightarrow C$  and a pointed simplicial subset  $Y \subset X$ . We use this to construct several spectral sequences for the calculation of higher topological Hochschild homology and apply those for calculations in some examples that could not be treated before.

## 1. INTRODUCTION

Considering relative versions of (co)homology theories is crucial for obtaining calculational and structural results. We work in the setting of commutative  $S$ -algebras (see [EKMM]) and these can be tensored with simplicial sets or topological spaces. Important invariants of a commutative  $S$ -algebra  $A$  are the homotopy groups of  $A \otimes X$  for suitable spaces or simplicial sets  $X$ . We call this the Loday construction of  $A$  with respect to  $X$  and denote it by  $\mathcal{L}_X(A)$ .

Important special cases are the higher order topological Hochschild homology of  $A$ ,

$$\mathrm{THH}_*^{[n]}(A) = \pi_*(A \otimes S^n),$$

where  $S^n$  is the  $n$ -sphere. For  $n = 1$  this reduces to ordinary topological Hochschild homology of  $A$ ,  $\mathrm{THH}(A)$ , which receives a trace map from the algebraic K-theory of  $A$ ,  $K(A)$ , and can be used via the construction of topological cyclic homology to obtain an approximation of  $K(A)$ . Torus homology of  $A$ ,  $\pi_*(A \otimes T^n)$  for the  $n$ -torus  $T^n$ , receives an  $n$ -fold iterated trace map from the iterated algebraic K-theory of  $A$ . Ongoing work by Ausoni and Dundas uses torus homology in order to make progress on the so-called red-shift conjecture [AuR].

One strength of the construction of  $A \otimes X$  is that it is functorial in both  $X$  and  $A$ , which allows us to study the homotopy type of  $A \otimes X$  by iteratively constructing  $X$  out of smaller spaces. This iterative method is for instance heavily used in Veen's work [V] and in [DLR], [BLPRZ]. As spheres are the building blocks of CW complexes, the calculation of  $\mathrm{THH}_*^{[n]}(A)$  is crucial for understanding  $\pi_*(\mathcal{L}_X(A))$  for CW-complexes  $X$ . The aim of this paper is to develop new tools for the calculation of higher order topological Hochschild homology by using the extra flexibility that is gained by a relative approach.

For a sequence of morphisms in the category of commutative  $S$ -algebras  $A \rightarrow B \rightarrow C$  and for a pair of pointed simplicial sets  $(X, Y)$  we consider a relative version of the Loday construction,  $\mathcal{L}_{(X,Y)}(A, B; C)$ : This relative version places  $C$  over the basepoint,  $B$  over all points of  $Y$  that are not the basepoint and  $A$  over the complement of  $Y$  in  $X$ .

We define this relative Loday construction and show some of its properties in Section 2. Section 3 exploits this relative structure and other geometric observations to establish several weak equivalences (juggling formulae) that compare higher THH groups with respect to the sphere spectrum as the ground ring and those with respect to other commutative  $S$ -algebra spectra as ground rings, and compare higher THH-groups  $\mathrm{THH}^{[m]}$  for  $m = n$  and  $n - 1$ . We

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use these juggling formulae to construct several spectral sequences for the calculation of higher topological Hochschild homology and apply these spectral sequences to obtain calculations for (higher) topological Hochschild homology that were not known before. In Section 4 we determine for instance higher order relative THH of certain Thom spectra and the higher order Shukla homology of  $\mathbb{F}_p$  with respect to pointed commutative  $\mathbb{F}_p$ -monoid algebras. We gain additive results about  $\mathrm{THH}^E(H\mathbb{F}_p)$  and  $\mathrm{THH}^{[2]}(E; H\mathbb{F}_p)$  for  $E = ko, ku, tmf$  at  $p = 2$  and  $E = \ell$  the Adams summand for  $p$  an odd prime. Furthermore, we show a splitting result for higher THH of the form  $\mathrm{THH}^{[n], Hk}(HA)$ , where  $k$  is any commutative ring and  $A$  is any commutative  $k$ -algebra.

In the following we work in the setting of [EKMM] and we use the model structure on commutative  $S$ -algebras from [EKMM, chapter VII]. Let  $A$  be a commutative  $S$ -algebra. As the category of commutative  $A$ -algebras is equivalent to the category of commutative  $S$ -algebras under  $A$ , we obtain a model category structure on the category of commutative  $A$ -algebras. In particular, a commutative  $A$ -algebra  $B$  is cofibrant if its unit map  $A \rightarrow B$  is a cofibration of commutative  $S$ -algebras.

## 2. THE RELATIVE LODAY CONSTRUCTION

Higher topological Hochschild homology of a commutative ring spectrum  $A$  is a special case of the Loday construction, or the internal tensor product, which sends  $A$  and a simplicial set  $X$  to a commutative simplicial ring spectrum  $\mathcal{L}_X(A) = X \otimes A$ , which is a commutative augmented  $A$ -algebra. This is a ring spectrum version of the Loday construction defined by Pirashvili in [P] for commutative rings, which sends a commutative ring  $R$  and a simplicial set  $X$  to the commutative augmented simplicial  $R$ -algebra  $X \otimes R$ .

For a cofibrant commutative  $S$ -algebra  $A$  this construction is homotopy invariant as a functor of  $X$ , that is: if one works with homotopy equivalent simplicial sets, we get homotopy equivalent augmented simplicial commutative  $A$ -algebras; in particular, this is true if one works with two simplicial models for the same space.

Let  $X$  be a pointed simplicial set. Since all boundary maps in a pointed simplicial set send the basepoint to the basepoint, given an  $A$ -module  $C$  (and in particular, a commutative  $A$ -algebra  $C$ ) we can also study Loday constructions with coefficients  $\mathcal{L}_X(A; C)$  which replaces the copy of  $A$  over the base point by a copy of  $C$ . We now define a relative version of this.

**Definition 2.1.** Let  $A$  be a commutative  $S$ -algebra,  $B$  a commutative  $A$ -algebra, and  $C$  a commutative  $B$ -algebra, with maps  $A \xrightarrow{f} B \xrightarrow{g} C$ . Let  $X$  be a pointed simplicial set and  $Y$  be a pointed simplicial subset. Then we can define

$$(\mathcal{L}_{(X,Y)}(A, B; C))_n := \left( \bigwedge_{X_n \setminus Y_n} A \right) \wedge \left( \bigwedge_{Y_n \setminus *} B \right) \wedge C.$$

We call this the  $n$ th simplicial degree of the relative Loday construction of  $A$  and  $B$  with coefficients in  $C$  on  $(X, Y)$ .

The smash product here is usually taken over the sphere spectrum, but can be done over any commutative ring spectrum,  $k$ . In this case, we will add a superscript to the notation,

$$\mathcal{L}_{(X,Y)}^k(A, B; C).$$

If  $B = C$  then the simplicial subset  $Y$  of  $X$  does not have to be pointed.

The structure maps of this construction use the fact that the smash product is the coproduct in the category of commutative  $S$ -algebras and they are given as follows: Let  $\varphi \in \Delta([m], [n])$  and let  $\varphi^*$  denote the induced map on  $X$  and  $Y$ :

$$\varphi^*: Y_n \rightarrow Y_m, \quad \varphi^*: X_n \rightarrow X_m.$$

Note that  $X_n \setminus Y_n$  is not a subcomplex of  $X_n$ , so  $\varphi^*$  might send elements in here to  $Y_m$ . We get an induced map

$$\varphi^*: \left( \bigwedge_{X_n \setminus Y_n} A \right) \wedge \left( \bigwedge_{Y_n \setminus *} B \right) \wedge C \rightarrow \left( \bigwedge_{X_m \setminus Y_m} A \right) \wedge \left( \bigwedge_{Y_m \setminus *} B \right) \wedge C$$

by first defining a map from

$$(2.1.1) \quad \left( \bigwedge_{X_n \setminus Y_n} A \right) \wedge \left( \bigwedge_{Y_n \setminus *} B \right) \wedge C$$

to a smash product over  $X_n$  where some of the smash factors  $A, B$  in (2.1.1) are sent to  $B$  or  $C$ :

- If for  $x \in X_n \setminus Y_n$  the image  $\varphi^*(x)$  is in  $Y_m \setminus *$ , then we apply the map  $f$  on the corresponding smash factor  $A$ .
- If  $x \in X_n \setminus Y_n$  is sent to the basepoint in  $Y_m$  under  $\varphi^*$ , then we use the composite  $g \circ f$  on the corresponding factor  $A$ .
- If for  $x \in X_n \setminus Y_n$ , the element  $\varphi^*(x)$  is in  $X_m \setminus Y_m$ , then we don't do anything, *i.e.*, we apply the identity map on the corresponding factor  $A$ .
- Similarly, if a  $y \in Y_n \setminus *$  is sent to the basepoint, then we apply  $g$  to the corresponding  $B$ -factor.
- If  $y \in Y_n \setminus *$  is sent to  $Y_m \setminus *$ , then we apply the identity map to the corresponding  $B$ -factor.
- By assumption, the basepoint is sent to the basepoint, so the  $C$ -factors are not involved in this process.

We now use the map  $\varphi$  to obtain a map to  $\left( \bigwedge_{X_m \setminus Y_m} A \right) \wedge \left( \bigwedge_{Y_m \setminus *} B \right) \wedge C$ :

- If an  $x \in X_m \setminus Y_m$  has multiple preimages under  $\varphi^*$ , then we use the multiplication on  $A$  on the corresponding smash factors.
- If a  $y \in Y_m \setminus *$  has multiple preimages under  $\varphi^*$ , then we use the multiplication on  $B$  on the corresponding smash factors.
- If the basepoint in  $Y_m$  has multiple preimages, then we apply the multiplication in  $C$  on the corresponding factors.
- If some  $x \in X_m \setminus Y_m$  is not in the image of  $\varphi^*$ , then we insert the unit map of  $A$  in the corresponding spots; similarly if  $y \in Y_m \setminus *$  has an empty preimage we use the unit map of  $B$ . The basepoint always has a non-empty preimage.

As the multiplication maps on  $A, B$  and  $C$  are associative and commutative and as the maps  $f$  and  $g$  are morphisms of commutative  $S$ -algebras, this gives the relative Loday construction the structure of a simplicial spectrum.

**Lemma 2.2.** *The relative Loday construction,  $\mathcal{L}_{(X,Y)}(A, B; C)_\bullet$ , is a simplicial augmented commutative  $C$ -algebra spectrum.*

*Proof.* The multiplication

$$\mathcal{L}_{(X,Y)}(A, B; C)_n \wedge_C \mathcal{L}_{(X,Y)}(A, B; C)_n \rightarrow \mathcal{L}_{(X,Y)}(A, B; C)_n$$

is defined coordinatewise and is therefore compatible with the simplicial structure. Hence we obtain a simplicial commutative augmented  $C$ -algebra structure on  $\mathcal{L}_{(X,Y)}(A, B; C)_\bullet$ .  $\square$

*Remark 2.3.* One can also define a version of the relative Loday construction if  $C$  is a  $B$ -module, rather than a commutative  $B$ -algebra. Then of course we cannot use the coproduct property of the smash product anymore, but we spelt out the structure maps explicitly, so that this generalization is not hard.

In this case  $\mathcal{L}_{(X,Y)}(A, B; C)_\bullet$  is a simplicial  $A$ -module spectrum and for the simplicial structure maps one has to use the  $B$ -module structure of  $C$ . Note that the commutative  $S$ -algebra  $A$  acts on  $C$  as well via the map  $f: A \rightarrow B$ .

**Example 2.4.** As an explicit example of a pointed simplicial subcomplex we consider  $\partial\Delta_2 \subset \Delta_2$  whose basepoint  $*$   $\in \Delta([n], [2])$  is the constant map with value 0. Note that the number of elements in  $\Delta([n], [m])$  is  $\binom{n+m+1}{n+1}$ .

We describe the effect of the maps  $\varphi: [1] \rightarrow [2]$ ,  $\varphi(0) = 0$ ,  $\varphi(1) = 2$  and  $\psi: [2] \rightarrow [1]$ ,  $\psi(1) = \psi(0) = 0$  and  $\psi(2) = 1$ .

$$\begin{array}{ccc} & & 2 \\ & \nearrow & \\ 1 & & 1 \\ \varphi \quad 0 & \longrightarrow & 0, \end{array} \quad \begin{array}{ccc} 2 & & \\ & \searrow & \\ & 1 & \\ \psi \quad 0 & \longrightarrow & 0 \end{array}$$

In  $\mathcal{L}_{(\Delta_2, \partial\Delta_2)}(A, B; C)_2$  there is only one copy of  $A$  because

$$\Delta_2[2] \setminus \partial\Delta_2([2]) = \Delta([2], [2]) \setminus \partial\Delta_2([2]) = \{\text{id}_{[2]}\}.$$

Thus

$$\mathcal{L}_{(\Delta_2, \partial\Delta_2)}(A, B; C)_2 = A \wedge \left( \bigwedge_{\partial\Delta_2([2]) \setminus \{\text{id}_{[2]}, *\}} B \right) \wedge C.$$

As  $\partial\Delta_2[1] = \Delta_2[1]$  we get

$$\mathcal{L}_{(\Delta_2, \partial\Delta_2)}(A, B; C)_1 = \left( \bigwedge_{\partial\Delta_2([1]) \setminus *} B \right) \wedge C.$$

The map  $\psi^*: \Delta_2([1]) \rightarrow \Delta_2([2])$  sends the six elements of  $\Delta_2([1])$  injectively to six elements in  $\Delta_2([2])$ , so on the Loday construction we only use the unit maps of  $A$  and  $B$  to fill in the gaps. In particular, the identity of  $[2]$  is not in the image of  $\psi$  and we get as  $\psi^*$  on the Loday construction

$$\begin{aligned} \left( \bigwedge_{\partial\Delta_2([1]) \setminus *} B \right) \wedge C &\xrightarrow{\cong} S \wedge \left( \bigwedge_{\Delta_2([2]) \setminus (\{\text{id}_{[2]}\} \cup \text{im}(\psi^*))} S \right) \wedge \left( \bigwedge_{\text{im}(\psi^*) \setminus *} B \right) \wedge C \\ \eta_A \wedge \left( \bigwedge_{\Delta_2([2]) \setminus (\{\text{id}_{[2]}\} \cup \text{im}(\psi^*))} \eta_B \right) \wedge \left( \bigwedge_{\text{im}(\psi^*) \setminus *} \text{id}_B \right) \wedge \text{id}_C &\downarrow \\ A \wedge \left( \bigwedge_{\partial\Delta_2([2]) \setminus \{\text{id}_{[2]}, *\}} B \right) &\wedge C. \end{aligned}$$

In contrast to this, the map  $\varphi^*: \Delta_2([2]) \rightarrow \Delta_2([1])$  is surjective. The preimage of the basepoint under  $\varphi^*$  is just the basepoint, so we get the identity on the  $C$ -factor in  $A \wedge \left( \bigwedge_{\partial\Delta_2([2]) \setminus \{\text{id}_{[2]}, *\}} B \right) \wedge C$ , but we have to use the map  $f: A \rightarrow B$  and several instances of the multiplication on  $B$  to get to  $\left( \bigwedge_{\partial\Delta_2([1]) \setminus *} B \right) \wedge C$  because there is a fiber of cardinality three and a fiber of cardinality two.

**Example 2.5.** If in Definition 2.1 we take  $A = B$ , then  $\mathcal{L}_{(X,Y)}(A, A; C)_\bullet = \mathcal{L}_X(A; C)_\bullet$ . If in addition  $A = C$  then we obtain  $\mathcal{L}_{(X,Y)}(A, A; A)_\bullet = \mathcal{L}_X(A)_\bullet$ .

**Example 2.6.** If we work relative to  $A$ , *i.e.*, if we consider  $\mathcal{L}_{(X,Y)}^A(A, B; C)_\bullet$ , then the  $A$ -factors disappear because we smash over  $A$  and we get

$$\mathcal{L}_{(X,Y)}^A(A, B; C)_\bullet \cong \mathcal{L}_Y^A(B; C)_\bullet.$$

**Definition 2.7.** We define *higher topological Hochschild homology of order  $n$  of  $A$  with coefficients in  $C$*  by  $\text{THH}^{[n],k}(A; C) := \mathcal{L}_{S^n}^k(A; C)_\bullet$ . Here,  $S^n$  is a pointed simplicial model of the  $n$ -sphere.

*Notation 2.8.* As above, if  $k$  is the sphere spectrum it is omitted from the notation. Similarly, if  $n = 1$  this may be omitted also and written as  $\mathrm{THH}^k(A; C)$ . If  $C = A$ , we may write  $\mathrm{THH}^{[n],k}(A)$ .

**Proposition 2.9.**

- (a) For  $A, B$ , and as in Definition 2.1,  $C$  a commutative  $B$ -algebra,  $X$  a pointed simplicial set and  $Y$  a pointed simplicial subset, we get an isomorphism of augmented simplicial commutative  $C$ -algebras

$$(2.9.1) \quad \mathcal{L}_{(X,Y)}^k(A, B; C)_\bullet \cong \mathcal{L}_X^k(A; C)_\bullet \wedge_{\mathcal{L}_Y^k(A; C)_\bullet} \mathcal{L}_Y^k(B; C)_\bullet.$$

- (b) For  $X_0$  a common pointed simplicial subset of  $X_1$  and  $X_2$  and  $Y_0$  a common pointed simplicial subset of  $Y_1$  and  $Y_2$  so that  $Y_i \subseteq X_i$  for  $i = 1, 2$  and  $Y_0 = X_0 \cap Y_1 \cap Y_2$ , we have an isomorphism of augmented simplicial  $C$ -algebras

$$(2.9.2) \quad \mathcal{L}_{(X_1 \cup_{X_0} X_2, Y_1 \cup_{Y_0} Y_2)}^k(A, B; C)_\bullet \cong \mathcal{L}_{(X_1, Y_1)}^k(A, B; C)_\bullet \wedge_{\mathcal{L}_{(X_0, Y_0)}^k(A, B; C)_\bullet} \mathcal{L}_{(X_2, Y_2)}^k(A, B; C)_\bullet.$$

If  $C = B$ , then in both statements we can work in the unpointed setting.

*Proof.* For the claim in (2.9.1) we have a levelwise isomorphism of simplicial spectra

$$(\mathcal{L}_X^k(A; C))_n \wedge_{(\mathcal{L}_Y^k(A; C))_n} (\mathcal{L}_Y^k(B; C))_n \cong (\mathcal{L}_{(X,Y)}^k(A, B; C))_n$$

given by the identification of coequalizers

$$\left( \bigwedge_{X_n \setminus Y_n} A \right) \wedge \left( \bigwedge_{Y_n \setminus *} A \right) \wedge C \bigwedge_{((\bigwedge_{Y_n \setminus *} A) \wedge C)} \left( \bigwedge_{Y_n \setminus *} B \right) \wedge C \cong \left( \bigwedge_{X_n \setminus Y_n} A \right) \wedge \left( \bigwedge_{Y_n \setminus *} B \right) \wedge C.$$

Similarly, for (2.9.2) we have a levelwise isomorphism of simplicial spectra

$$(\mathcal{L}_{(X_1, Y_1)}^k(A, B; C))_n \wedge_{(\mathcal{L}_{(X_0, Y_0)}^k(A, B; C))_n} (\mathcal{L}_{(X_2, Y_2)}^k(A, B; C))_n \cong (\mathcal{L}_{(X_1 \cup_{X_0} X_2, Y_1 \cup_{Y_0} Y_2)}^k(A, B; C))_n.$$

Here we use that tensoring a commutative  $S$ -algebra with a pointed simplicial set is compatible with pushouts of simplicial sets, hence we get

$$\begin{aligned} \left( \bigwedge_{(X_1)_n \setminus (Y_1)_n} A \wedge \bigwedge_{(Y_1)_n \setminus *} B \wedge C \right) \bigwedge_{(\bigwedge_{(X_0)_n \setminus (Y_0)_n} A \wedge \bigwedge_{(Y_0)_n \setminus *} B \wedge C)} & \left( \bigwedge_{(X_2)_n \setminus (Y_2)_n} A \wedge \bigwedge_{(Y_2)_n \setminus *} B \wedge C \right) \\ \cong \bigwedge_{(X_1 \cup_{X_0} X_2)_n \setminus (Y_1 \cup_{Y_0} Y_2)_n} A \wedge \bigwedge_{y \in (Y_1 \cup_{Y_0} Y_2)_n \setminus *} B \wedge C. \end{aligned}$$

□

Let  $A$  be an augmented commutative  $C$ -algebra, i.e., in addition to the map  $g \circ f: A \rightarrow C$  we have a map  $\eta: C \rightarrow A$ , such that  $g \circ f \circ \eta = \mathrm{id}_C$ . In that case, we can identify the relative Loday construction  $\mathcal{L}_{(X,Y)}^C(A, C; C)_\bullet$  with the Loday construction of the quotient:

**Proposition 2.10.** *Let  $A$  be an augmented commutative  $C$ -algebra. Then there is an isomorphism of simplicial augmented commutative  $C$ -algebras*

$$(2.10.1) \quad \mathcal{L}_{(X,Y)}^C(A, C; C)_\bullet \cong \mathcal{L}_{X/Y}^C(A; C)_\bullet.$$

where  $X/Y$  has the equivalence class of  $Y$  as a basepoint.

*Proof.* We use Proposition 2.9 above and obtain that

$$\mathcal{L}_{(X,Y)}^C(A, C; C)_\bullet \cong \mathcal{L}_X^C(A; C)_\bullet \wedge_{\mathcal{L}_Y^C(A; C)_\bullet} \mathcal{L}_Y^C(C; C)_\bullet.$$

but  $\mathcal{L}_Y^C(C; C)_\bullet$  is isomorphic to the constant simplicial object  $C_\bullet$  with  $C$  in every simplicial degree. Thus

$$\mathcal{L}_{(X,Y)}^C(A, C; C)_\bullet \cong \mathcal{L}_X^C(A; C)_\bullet \wedge_{\mathcal{L}_Y^C(A; C)_\bullet} C_\bullet \cong \mathcal{L}_{X/Y}^C(A; C)_\bullet \wedge_C C \cong \mathcal{L}_{X/Y}^C(A; C)_\bullet.$$

as claimed.  $\square$

Proposition 2.10 immediately gives rise to the following spectral sequence.

**Proposition 2.11.** *If  $C$  is a cofibrant commutative  $S$ -algebra and  $A$  is a cofibrant commutative augmented  $C$ -algebra and if  $Y$  is a pointed simplicial subset of  $X$ , then there is a spectral sequence*

$$E_{s,t}^2 = \mathrm{Tor}_{s,t}^{\pi_*(\mathcal{L}_Y^C(A;C)_\bullet)}(\pi_*(\mathcal{L}_X^C(A;C)_\bullet), \pi_*C) \Rightarrow \pi_*(\mathcal{L}_{X/Y}^C(A;C)_\bullet).$$

*Proof.* The isomorphism from Proposition 2.10

$$\mathcal{L}_X^C(A;C)_\bullet \wedge_{\mathcal{L}_Y^C(A;C)_\bullet} C \cong \mathcal{L}_{X/Y}^C(A;C)_\bullet$$

induces a weak equivalence

$$\mathcal{L}_X^C(A;C)_\bullet \wedge_{\mathcal{L}_Y^C(A;C)_\bullet}^L C \sim \mathcal{L}_{X/Y}^C(A;C)_\bullet$$

and we get the associated Künneth spectral sequence.  $\square$

### 3. SPECTRAL SEQUENCES WITH THE RELATIVE LODAY CONSTRUCTION

In this section we set up some spectral sequences. Let  $S$  be the sphere spectrum and let  $R$  be a commutative  $S$ -algebra. Unadorned smash products will be over  $S$ . We first recall some properties of the category of commutative  $R$ -algebras: The category of commutative  $R$ -algebras is a topological model category ([EKMM, VII.4.10]). This implies that it is tensored over the category of unbased spaces and that for every sequence of cofibrations  $R \rightarrow A \rightarrow B$  of commutative  $S$ -algebras and every relative CW-complex  $(L, K)$  the map

$$(A \otimes L) \wedge_{(A \otimes K)} (B \otimes K) \rightarrow B \otimes L$$

is a cofibration. For a simplicial finite set  $X$  and a commutative  $R$ -algebra  $A$  there is a natural isomorphism (see [EKMM, VII.3.2]):

$$|\mathcal{L}_X^R(A)_\bullet| \cong A \otimes |X|.$$

We define the Loday construction  $\mathcal{L}_{|X|}^R(A)$  as  $A \otimes |X|$ . We get a similar definition in the pointed setting and we can define the relative Loday construction for a pair of pointed CW complexes  $Y \subset X$  and a sequence of maps of commutative  $S$ -algebras  $R \rightarrow A \rightarrow B \rightarrow C$  using Proposition 2.9 (a) as

$$(3.0.1) \quad \mathcal{L}_{(X,Y)}^R(A, B; C) := \mathcal{L}_X^R(A; C) \wedge_{\mathcal{L}_Y^R(A; C)}^L \mathcal{L}_Y^R(B; C).$$

**Theorem 3.1.** *Let  $A$  be a cofibrant commutative  $S$ -algebra, and let  $B$  be a cofibrant commutative  $A$ -algebra. There is an equivalence of augmented commutative  $B$ -algebras*

$$\mathcal{L}_{(D^n, S^{n-1})}(A, B) \simeq \mathrm{THH}^{[n-1], A}(B)$$

for all  $n$ .

*Proof.* We proceed by induction on  $n$ . For  $n = 1$ ,  $\mathcal{L}_{(D^1, S^0)}(A, B)$  is the two-sided bar construction  $B(B, A, B)$  which is a model for  $B \wedge_A^L B$ . As we assumed  $B$  to be a cofibrant commutative  $A$ -algebra  $B \wedge_A^L B$  is weakly equivalent to  $B \wedge_A B$  which is  $\mathrm{THH}^{[0], A}(B)$ . For the inductive step, we assume that  $\mathcal{L}_{(D^n, S^{n-1})}(A, B) \simeq \mathrm{THH}^{[n-1], A}(B)$ . By [V], we know that  $\mathrm{THH}^{[n], A}(B)$  is weakly equivalent to the bar construction  $B^A(B, \mathrm{THH}^{[n-1], A}(B), B)$  by decomposing the  $n$ -sphere into two hemispheres glued along an  $(n-1)$ -sphere.

We also know that  $\mathcal{L}_{(D^{n+1}, S^n)}(A, B)$  can be built from two half-disks of dimension  $n+1$ , part of whose boundary (the outside edge) has  $B$ 's over it, and the other part (the  $n$ -disk we glue along) has  $A$ 's over it. So by (2.9.2),

$$\begin{aligned} \mathcal{L}_{(D^{n+1}, S^n)}(A, B) &= \mathcal{L}_{(D^{n+1} \cup_{D^n} D^{n+1}, D^n \cup_{S^{n-1}} D^n)}(A, B) \\ &\cong \mathcal{L}_{(D^{n+1}, D^n)}(A, B) \wedge_{\mathcal{L}_{(D^n, S^{n-1})}(A, B)} \mathcal{L}_{(D^{n+1}, D^n)}(A, B). \end{aligned}$$

For example, when  $n = 1$  we have

$$\mathcal{L}_{(D^2, S^1)}(A, B) \simeq \mathcal{L}\left(\begin{array}{c} \bullet \\ \circlearrowleft \end{array}\right)^{(A, B)} \wedge_{\mathcal{L}\left(\begin{array}{c} | \\ \cdot \end{array}\right)^{(A, B)}} \mathcal{L}\left(\begin{array}{c} \bullet \\ \circlearrowright \end{array}\right)^{(A, B)}$$

So we have

$$\begin{aligned} \mathcal{L}_{(D^{n+1}, S^n)}(A, B) &\cong \mathcal{L}_{(D^{n+1}, D^n)}(A, B) \wedge_{\mathcal{L}_{(D^n, S^{n-1})}(A, B)} \mathcal{L}_{(D^{n+1}, D^n)}(A, B) \\ &\simeq \mathcal{L}_{(*, *)}(A, B) \wedge_{\mathcal{L}_{(D^n, S^{n-1})}(A, B)}^L \mathcal{L}_{(*, *)}(A, B) && \text{(by homotopy invariance)} \\ &\simeq B \wedge_{\mathcal{L}_{(D^n, S^{n-1})}(A, B)}^L B \\ &\simeq B^A(B, \mathcal{L}_{(D^n, S^{n-1})}(A, B), B) \\ &\simeq B^A(B, \mathrm{THH}^{[n-1], A}(B), B) && \text{(by assumption)} \\ &\simeq \mathrm{THH}^{[n], A}(B). && \text{(by [V])} \end{aligned}$$

□

Let  $C$  be a commutative  $R$ -algebra. Let  $\mathcal{C}_{R/C}$  and  $\mathcal{C}_{C/C}$  denote the categories of commutative  $R$ -algebras over  $C$  and of commutative  $C$ -algebras over  $C$ . Let  $\mathcal{T}$  denote the category of based spaces. We then have a functor

$$\bar{\otimes}_C: \mathcal{C}_{R/C} \times \mathcal{T} \rightarrow \mathcal{C}_{C/C}$$

defined by  $(A, X) \mapsto A \bar{\otimes}_C X := (A \otimes X) \wedge_A C$ . Here, the map  $A \rightarrow A \otimes X$  is given by the composition of the isomorphism  $A \cong A \otimes *$  with the map induced by the inclusion of the basepoint. The augmentation  $A \bar{\otimes}_C X \rightarrow C$  is given by

$$(A \otimes X) \wedge_A C \rightarrow (A \otimes *) \otimes_A C \cong C.$$

We have a natural homeomorphism

$$\mathcal{C}_{C/C}(A \bar{\otimes}_C X, B) \cong \mathcal{T}(X, \mathcal{C}_{R/C}(A, B)).$$

Let  $D \rightarrow E$  be a map in  $\mathcal{C}_{R/C}$  such that the underlying map of commutative  $R$ -algebras is a cofibration. Let  $K \rightarrow L$  be an inclusion of based spaces such that  $(L, K)$  is a relative  $CW$ -complex. Then the natural map

$$(D \bar{\otimes}_C L) \wedge_{(D \bar{\otimes}_C K)} (E \bar{\otimes}_C K) \rightarrow E \bar{\otimes}_C L$$

is a cofibration of commutative  $R$ -algebras. For  $A \in \mathcal{C}_{R/C}$  and a simplicial finite pointed set  $X$ , we have a natural isomorphism of  $C$ -algebras over  $C$ :

$$|\mathcal{L}_{(X, *)}^R(A; C)_\bullet| \cong A \bar{\otimes}_C |X|.$$

**Theorem 3.2.** *Let  $S \rightarrow A \rightarrow B \rightarrow C$  be a sequence of cofibrations of commutative  $S$ -algebras. Then*

- (a)  $\mathrm{THH}^{[n], A}(B) \cong A \wedge_{\mathrm{THH}^{[n]}(A)} \mathrm{THH}^{[n]}(B)$  and
- (b)  $\mathrm{THH}^{[n], A}(B; C) \cong C \wedge_{\mathrm{THH}^{[n]}(A; C)} \mathrm{THH}^{[n]}(B; C).$

*In both cases, the smash product represents the derived smash product.*

*Proof.* In order to show (a) we first prove that

$$A \wedge_{\mathrm{THH}^{[n]}(A)} \mathrm{THH}^{[n]}(B)$$

represents the derived smash product of  $A$  and  $\mathrm{THH}^{[n]}(B)$  over  $\mathrm{THH}^{[n]}(A)$ . For this we first show that  $\mathrm{THH}^{[n]}(A)$  is a cofibrant commutative  $S$ -algebra: Since  $A$  is a cofibrant commutative  $S$ -algebra, it suffices to show that the unit  $A \rightarrow \mathrm{THH}^{[n]}(A)$  or equivalently that the map

$A \otimes * \rightarrow A \otimes S^n$  is a cofibration of commutative  $S$ -algebras. By the properties listed above the map

$$(S \otimes S^n) \wedge_{(S \otimes *)} (A \otimes *) \rightarrow (A \otimes S^n)$$

is a cofibration. The map  $S \otimes * \rightarrow S \otimes S^n$  is an isomorphism because both sides can be identified with  $S$ . We get that

$$(A \otimes *) \cong (S \otimes S^n) \wedge_{(S \otimes *)} (A \otimes *).$$

Thus,  $\mathrm{THH}^{[n]}(A)$  is cofibrant. We now show that  $\mathrm{THH}^{[n]}(A) \rightarrow \mathrm{THH}^{[n]}(B)$  is a cofibration of commutative  $S$ -algebras. For this it suffices to show that  $A \otimes S^n \rightarrow B \otimes S^n$  is a cofibration. Since  $A \rightarrow B$  is a cofibration, the map

$$(A \otimes S^n) \otimes_{(A \otimes *)} (B \otimes *) \rightarrow (B \otimes S^n)$$

is a cofibration. The map  $A \otimes * \rightarrow B \otimes *$  is a cofibration because it can be identified with the map  $A \rightarrow B$ . Because cofibrations are stable under cobase change the map

$$(A \otimes S^n) \rightarrow (A \otimes S^n) \otimes_{(A \otimes *)} (B \otimes *)$$

is a cofibration. Thus  $A \otimes S^n \rightarrow B \otimes S^n$  is a cofibration. Because  $\mathrm{THH}^{[n]}(A) \rightarrow \mathrm{THH}^{[n]}(B)$  is a cofibration between cofibrant commutative  $S$ -algebras, we get by [EKMM, VII.7.4] that the functor

$$- \wedge_{\mathrm{THH}^{[n]}(A)} \mathrm{THH}^{[n]}(B)$$

preserves weak equivalences between cofibrant commutative  $S$ -algebras. We factor the map  $\mathrm{THH}^{[n]}(A) \rightarrow A$  as a cofibration followed by an acyclic fibration

$$\mathrm{THH}^{[n]}(A) \hookrightarrow \tilde{A} \xrightarrow{\sim} A$$

and obtain a weak equivalence

$$\tilde{A} \wedge_{\mathrm{THH}^{[n]}(A)} \mathrm{THH}^{[n]}(B) \xrightarrow{\sim} A \wedge_{\mathrm{THH}^{[n]}(A)} \mathrm{THH}^{[n]}(B).$$

By [EKMM, VII.6.7] the  $S$ -algebra

$$\tilde{A} \wedge_{\mathrm{THH}^{[n]}(A)} \mathrm{THH}^{[n]}(B)$$

represents the derived smash product of  $A$  and  $\mathrm{THH}^{[n]}(B)$  over  $\mathrm{THH}^{[n]}(A)$ .

We now show that there is an isomorphism of commutative  $S$ -algebras

$$\mathrm{THH}^{[n],A}(B) \cong A \wedge_{\mathrm{THH}^{[n]}(A)} \mathrm{THH}^{[n]}(B).$$

We start with the isomorphism of commutative  $S$ -algebras

$$\begin{aligned} A \wedge_{\mathrm{THH}^{[n]}(A)} \mathrm{THH}^{[n]}(B) &\cong |\mathcal{L}_*(A)_\bullet| \wedge_{|\mathcal{L}_{S^n(A)}_\bullet|} |\mathcal{L}_{S^n(B)}_\bullet| \\ &\cong |\mathcal{L}_*(A)_\bullet \wedge_{\mathcal{L}_{S^n(A)}_\bullet} \mathcal{L}_{S^n(B)}_\bullet|. \end{aligned}$$

By a comparison of coequalizer diagrams we have, for all  $n$ , isomorphisms of commutative  $S$ -algebras:

$$A \wedge_{A^{\wedge n}} (B^{\wedge n}) \cong \underbrace{B \wedge_A \dots \wedge_A B}_n$$

and these induce an isomorphism of simplicial commutative  $S$ -algebras

$$\mathcal{L}_*(A)_\bullet \wedge_{\mathcal{L}_{S^n(A)}_\bullet} \mathcal{L}_{S^n(B)}_\bullet \cong \mathcal{L}_{S^n(B)}^A_\bullet.$$

This proves part (a) of the theorem.

We now prove part (b). We again first show that

$$C \wedge_{\mathrm{THH}^{[n]}(A;C)} \mathrm{THH}^{[n]}(B;C)$$

represents the derived smash product of  $C$  and  $\mathrm{THH}^{[n]}(B;C)$  over  $\mathrm{THH}^{[n]}(A;C)$ . For this it suffices to show that  $\mathrm{THH}^{[n]}(A;C)$  is a cofibrant commutative  $S$ -algebra and that the map



$\mathrm{THH}^{[n]}(A; C) \rightarrow \mathrm{THH}^{[n]}(B; C)$  is a cofibration of commutative  $S$ -algebras. The morphism  $C \rightarrow \mathrm{THH}^{[n]}(A; C)$  is a cofibration because

$$C \rightarrow A \bar{\otimes}_C S^n = (A \otimes S^n) \wedge_A C$$

is a cofibration. Thus  $\mathrm{THH}^{[n]}(A; C)$  is cofibrant. The map  $\mathrm{THH}^{[n]}(A; C) \rightarrow \mathrm{THH}^{[n]}(B; C)$  is a cofibration because  $A \bar{\otimes}_C S^n \rightarrow B \bar{\otimes}_C S^n$  can be written as

$$A \bar{\otimes}_C S^n \rightarrow (A \bar{\otimes}_C S^n) \wedge_{(A \bar{\otimes}_C *)} (B \bar{\otimes}_C *) \rightarrow B \bar{\otimes}_C S^n.$$

The first map of the composition is an isomorphism, because the map  $A \bar{\otimes}_C * \rightarrow B \bar{\otimes}_C *$  identifies with the identity of  $C$ .

It remains to prove that there is an isomorphism of commutative  $S$ -algebras

$$\mathrm{THH}^{[n], A}(B; C) \cong C \wedge_{\mathrm{THH}^{[n]}(A; C)} \mathrm{THH}^{[n]}(B; C).$$

This follows as above by using that we have an isomorphism of commutative  $S$ -algebras

$$C \wedge_{(A^{\wedge n} \wedge C)} (B^{\wedge n} \wedge C) \cong B^{\wedge n} C$$

for all  $n \geq 0$ . □

*Remark 3.3.* The proof shows that Theorem 3.2 also holds for general finite pointed simplicial sets  $X$  and a sequence of cofibrations of commutative  $S$ -algebras  $S \rightarrow A \rightarrow B \rightarrow C$ , giving us isomorphisms

- (a)  $\mathcal{L}_{|X|}^A(B) \cong A \wedge_{\mathcal{L}_{|X|}(A)} \mathcal{L}_{|X|}(B)$  and
- (b)  $\mathcal{L}_{|X|}^A(B; C) \cong C \wedge_{\mathcal{L}_{|X|}(A; C)} \mathcal{L}_{|X|}(B; C).$

*Remark 3.4.* It is known that for topological Hochschild homology, there is a difference between Galois descent and étale descent: John Rognes [R] developed the notion of Galois extensions for commutative  $S$ -algebras and showed that for a Galois extension  $A \rightarrow B$  with finite Galois group  $G$  the canonical map  $B \rightarrow \mathrm{THH}^A(B)$  is a weak equivalence [R, Lemma 9.2.6]. Akhil Mathew [M] provided an example of such a Galois extension that does *not* satisfy étale descent, *i.e.*, the pushout map

$$B \wedge_A \mathrm{THH}(A) \rightarrow \mathrm{THH}(B)$$

is not a weak equivalence. Theorem 3.2 doesn't contradict this. We take a finite Galois extension  $A \rightarrow B$ . Then we obtain a weak equivalences

$$B \rightarrow \mathrm{THH}^A(B) \cong A \wedge_{\mathrm{THH}(A)} \mathrm{THH}(B).$$

But if we then smash this equivalence with  $\mathrm{THH}(A)$  over  $A$  the resulting equivalence

$$(3.4.1) \quad B \wedge_A \mathrm{THH}(A) \simeq (A \wedge_{\mathrm{THH}(A)} \mathrm{THH}(B)) \wedge_A \mathrm{THH}(A)$$

cannot be reduced to the statement that  $B \wedge_A \mathrm{THH}(A)$  is equivalent to  $\mathrm{THH}(B)$ : On the right hand side of (3.4.1) we cannot reduce the  $\mathrm{THH}(A)$ -term because in the smash product we use the augmentation map  $\mathrm{THH}(A) \rightarrow A$  and its composite with the unit is not equivalent to the identity map.

Let  $R$  be a commutative  $S$ -algebra, and  $\mathcal{C}_R$  the category of commutative  $R$ -algebras. Let  $D$  be the category  $\{b \leftarrow a \rightarrow c\}$ . Then the category  ${}^D\mathcal{C}_R$  of functors from  $D$  to  $\mathcal{C}_R$  admits a model category structure, where the weak equivalences (resp. fibrations) are the maps that are objectwise weak equivalences (resp. fibrations). We have a cofibrant replacement functor  ${}^D\mathcal{C}_R \rightarrow {}^D\mathcal{C}_R$ . The homotopy pushout  $B \tilde{\wedge}_A C$  of a diagram  $B \leftarrow A \rightarrow C$  in  $\mathcal{C}_R$  is constructed by taking the chosen cofibrant replacement  $B' \leftarrow A' \rightarrow C'$  of the diagram and then taking the usual pushout  $B' \wedge_{A'} C'$ . One gets a functor

$$(-) \tilde{\wedge}_{(-)} (-): {}^D\mathcal{C}_R \rightarrow \mathcal{C}_R.$$

This functor sends weak equivalences to weak equivalences. There is natural map

$$B \tilde{\wedge}_A C \rightarrow B \wedge_A C$$

which is a weak equivalence when  $A$  is cofibrant and  $A \rightarrow B$  and  $A \rightarrow C$  are cofibrations. If  $A$  is cofibrant, then  $B \tilde{\wedge}_A C$  is equivalent to the derived smash product  $B \wedge_A^L C$  of  $B$  and  $C$  over  $A$ . One can show:

**Lemma 3.5.** *For a commutative diagram*

$$\begin{array}{ccccc} E & \longleftarrow & D & \longrightarrow & F \\ \uparrow & & \uparrow & & \uparrow \\ B & \longleftarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ H & \longleftarrow & G & \longrightarrow & I \end{array}$$

in  $\mathcal{C}_R$  there is a zig-zag of weak equivalences

$$(E \tilde{\wedge}_D F) \tilde{\wedge}_{(B \tilde{\wedge}_A C)} (H \tilde{\wedge}_G I) \sim (E \tilde{\wedge}_B H) \tilde{\wedge}_{(D \tilde{\wedge}_A G)} (F \tilde{\wedge}_C I)$$

over  $(E \wedge_B H) \wedge_{(D \wedge_A G)} (F \wedge_C I)$  where

$$(E \tilde{\wedge}_D F) \tilde{\wedge}_{(B \tilde{\wedge}_A C)} (H \tilde{\wedge}_G I) \rightarrow (E \wedge_B H) \wedge_{(D \wedge_A G)} (F \wedge_C I)$$

is given by the composition of the morphism

$$(E \tilde{\wedge}_D F) \tilde{\wedge}_{(B \tilde{\wedge}_A C)} (H \tilde{\wedge}_G I) \rightarrow (E \wedge_D F) \tilde{\wedge}_{(B \wedge_A C)} (H \wedge_G I) \rightarrow (E \wedge_D F) \wedge_{(B \wedge_A C)} (H \wedge_G I)$$

with the standard isomorphism

$$(E \wedge_D F) \wedge_{(B \wedge_A C)} (H \wedge_G I) \cong (E \wedge_B H) \wedge_{(D \wedge_A G)} (F \wedge_C I)$$

and

$$(E \tilde{\wedge}_B H) \tilde{\wedge}_{(D \tilde{\wedge}_A G)} (F \tilde{\wedge}_C I) \rightarrow (E \wedge_B H) \wedge_{(D \wedge_A G)} (F \wedge_C I)$$

is given by

$$(E \tilde{\wedge}_B H) \tilde{\wedge}_{(D \tilde{\wedge}_A G)} (F \tilde{\wedge}_C I) \rightarrow (E \wedge_B H) \tilde{\wedge}_{(D \wedge_A G)} (F \wedge_C I) \rightarrow (E \wedge_B H) \wedge_{(D \wedge_A G)} (F \wedge_C I).$$

**Theorem 3.6.** *Let  $S \rightarrow A \rightarrow B \rightarrow C$  be a sequence of cofibrations of commutative  $S$ -algebras. Then*

$$\mathrm{THH}^{[n]}(B; C) \sim \mathrm{THH}^{[n]}(A; C) \wedge_{\mathrm{THH}^{[n-1], A}(C)}^L \mathrm{THH}^{[n-1], B}(C).$$

*Proof.* We work in the model category of commutative  $S$ -algebras. For a map of commutative  $S$ -algebras  $D \rightarrow E$  we define commutative  $S$ -algebras  $T^{[n], D}(E)$  augmented over  $E$  inductively as follows: Let  $T^{[0], D}(E)$  be  $E \tilde{\wedge}_D E$  and let  $T^{[0], D}(E) \rightarrow E$  be defined by

$$E \tilde{\wedge}_D E \rightarrow E \wedge_D E \rightarrow E.$$

Set  $T^{[n+1], D}(E) := E \tilde{\wedge}_{T^{[n], D}(E)} E$  and define  $T^{[n+1], D}(E) \rightarrow E$  by

$$E \tilde{\wedge}_{T^{[n], D}(E)} E \rightarrow E \wedge_{T^{[n], D}(E)} E \rightarrow E.$$

The  $T^{[n], (-)}(E)$  are then endofunctors on the category of commutative  $S$ -algebras over  $E$ . Using the decomposition  $S^n = D^n \cup_{S^{n-1}} D^n$ , one can show that there are zig-zags of weak equivalences over  $C$  (compare [V])

$$\begin{aligned} \mathrm{THH}^{[n], A}(C) &\sim C \tilde{\wedge}_{\mathrm{THH}^{[n-1], A}(C)} C \\ \mathrm{THH}^{[n]}(A; C) &\sim C \tilde{\wedge}_{\mathrm{THH}^{[n-1]}(A; C)} C. \end{aligned}$$

With that it follows that there are equivalences over  $C$

$$\begin{aligned} \mathrm{THH}^{[n], A}(C) &\sim T^{[n], A}(C) \\ \mathrm{THH}^{[n]}(A; C) &\sim T^{[n-1], C \wedge_S A}(C). \end{aligned}$$

The same is true for  $B$  instead of  $A$ .

It thus suffices to show:

$$T^{[n], C \wedge_S B}(C) \sim T^{[n], C \wedge_S A}(C) \tilde{\wedge}_{T^{[n], A}(C)} T^{[n], B}(C).$$

We prove by induction on  $n$  that these  $S$ -algebras are equivalent via a zig-zag of weak equivalences over  $C$  where the augmentation of the right-hand side is given by

$$T^{[n], C \wedge_S A}(C) \tilde{\wedge}_{T^{[n], A}(C)} T^{[n], B}(C) \rightarrow T^{[n], C \wedge_S A}(C) \wedge_{T^{[n], A}(C)} T^{[n], B}(C) \rightarrow C.$$

We have an isomorphism  $T^{[0], C \wedge_S B}(C) \cong T^{[0], (C \wedge_S A) \wedge_A B}(C)$  over  $C$ . Because of the cofibrancy assumptions the map

$$(C \wedge_S A) \tilde{\wedge}_A B \rightarrow (C \wedge_S A) \wedge_A B$$

is a weak equivalence. It induces a weak equivalence

$$(C \tilde{\wedge}_C C) \tilde{\wedge}_{((C \wedge_S A) \tilde{\wedge}_A B)} (C \tilde{\wedge}_C C) \xrightarrow{\sim} C \tilde{\wedge}_{((C \wedge_S A) \wedge_A B)} C = T^{[0], (C \wedge_S A) \wedge_A B}(C).$$

This is a map over  $C$  if we endow the left-hand side with the augmentation

$$(C \tilde{\wedge}_C C) \tilde{\wedge}_{((C \wedge_S A) \tilde{\wedge}_A B)} (C \tilde{\wedge}_C C) \rightarrow (C \tilde{\wedge}_C C) \wedge_{((C \wedge_S A) \tilde{\wedge}_A B)} (C \tilde{\wedge}_C C) \rightarrow C \tilde{\wedge}_C C \rightarrow C.$$

By Lemma 3.5, we have an equivalence

$$(C \tilde{\wedge}_C C) \tilde{\wedge}_{((C \wedge_S A) \tilde{\wedge}_A B)} (C \tilde{\wedge}_C C) \sim (C \tilde{\wedge}_{(C \wedge_S A)} C) \tilde{\wedge}_{(C \tilde{\wedge}_A C)} (C \tilde{\wedge}_B C)$$

and the right-hand side is equal to  $T^{[0], C \wedge_S A}(C) \tilde{\wedge}_{T^{[0], A}(C)} T^{[0], B}(C)$ . The compatibility of the equivalence with the isomorphism

$$(C \wedge_C C) \wedge_{((C \wedge_S A) \wedge_A B)} (C \wedge_C C) \cong (C \wedge_{(C \wedge_S A)} C) \wedge_{(C \wedge_A C)} (C \wedge_B C)$$

implies that it is an equivalence over  $C$ .

We now assume that the claim is true for  $n$ . Set  $T' = T^{[n], C \wedge_S A}(C) \tilde{\wedge}_{T^{[n], A}(C)} T^{[n], B}(C)$ . By induction hypothesis we have

$$T^{[n+1], C \wedge_S B}(C) = C \tilde{\wedge}_{T^{[n], C \wedge_S B}(C)} C \sim C \tilde{\wedge}_{T'} C.$$

via a zig-zag of weak equivalences over  $C$ . We have a weak equivalence

$$(C \tilde{\wedge}_C C) \tilde{\wedge}_{T'} (C \tilde{\wedge}_C C) \xrightarrow{\sim} C \tilde{\wedge}_{T'} C.$$

It is a map over  $C$  if we endow the right-hand side with the augmentation

$$(C \tilde{\wedge}_C C) \tilde{\wedge}_{T'} (C \tilde{\wedge}_C C) \rightarrow (C \tilde{\wedge}_C C) \wedge_{T'} (C \tilde{\wedge}_C C) \rightarrow C \tilde{\wedge}_C C \rightarrow C.$$

By Lemma 3.5 we have an equivalence

$$(C \tilde{\wedge}_C C) \tilde{\wedge}_{T'} (C \tilde{\wedge}_C C) \sim (C \tilde{\wedge}_{T^{[n], C \wedge_S A}(C)} C) \tilde{\wedge}_{(C \tilde{\wedge}_{T^{[n], A}(C)} C)} (C \tilde{\wedge}_{T^{[n], B}(C)} C)$$

and the right-hand side is equal to  $T^{[n+1], C \wedge_S A}(C) \tilde{\wedge}_{T^{[n+1], A}(C)} T^{[n+1], B}(C)$ . Because of the compatibility with the isomorphism

$$\begin{aligned} & (C \wedge_C C) \wedge_{(T^{[n], C \wedge_S A}(C) \wedge_{T^{[n], A}(C)} T^{[n], B}(C))} (C \wedge_C C) \\ & \cong (C \wedge_{T^{[n], C \wedge_S A}(C)} C) \wedge_{(C \wedge_{T^{[n], A}(C)} C)} (C \wedge_{T^{[n], B}(C)} C) \end{aligned}$$

it is an equivalence over  $C$ . This shows the induction step.  $\square$

## 4. APPLICATIONS

### 4.1. Thom spectra.

**Example 4.1.** Christian Schlichtkrull [Sch] gives a general formula for the Loday construction on Thom spectra  $\mathcal{L}_X(T(f); M)$  where  $f: A \rightarrow BF_{h\mathcal{I}}$  is an  $E_\infty$ -map,  $A$  is a grouplike  $E_\infty$ -space, and  $BF_{h\mathcal{I}}$  is a model for  $BF = BGL_1(S)$ , the classifying space for stable spherical fibrations. The commutative  $S$ -algebra  $T(f)$  is the associated Thom spectrum for  $f$  and  $M$  is any  $T(f)$ -module.

If we set  $B = C$  in Theorem 3.2, then we obtain

$$(4.1.1) \quad \mathrm{THH}^A(B) \simeq B \wedge_{\mathrm{THH}^{[n]}(A;B)}^L \mathrm{THH}^{[n]}(B)$$

so if there is a factorization as follows

$$\begin{array}{ccc} A & \xrightarrow{f} & BF_{h\mathcal{I}} \\ h \downarrow & \nearrow g & \\ B & & \end{array}$$

such that  $h$  is a map of grouplike  $E_\infty$ -spaces, then we get an induced map of commutative  $S$ -algebras  $T(f) \rightarrow T(g)$ .

For  $X$  a sphere and  $M = T(g)$ , we obtain [Sch, Theorem 1]

$$\mathrm{THH}^{[n]}(T(f); T(g)) \simeq T(g) \wedge \Omega^\infty(S^n \wedge \mathbb{A})_+$$

where  $\mathbb{A}$  denotes the spectrum associated to  $A$  such that the map from  $A$  to the underlying infinite loop space of  $\mathbb{A}$ ,  $\Omega^\infty \mathbb{A}$ , is a weak equivalence.

Our juggling formula (4.1.1) gives a formula for higher  $\mathrm{THH}$  of  $T(g)$  as a commutative  $T(f)$ -algebra:

$$\begin{aligned} \mathrm{THH}^{[n], T(f)}(T(g)) &\simeq T(g) \wedge_{\mathrm{THH}^{[n]}(T(f); T(g))}^L \mathrm{THH}^{[n]}(T(g)) \\ &\simeq T(g) \wedge_{T(g) \wedge \Omega^\infty(S^n \wedge \mathbb{A})_+}^L T(g) \wedge \Omega^\infty(S^n \wedge \mathbb{B})_+. \end{aligned}$$

Important examples of such factorizations are listed for instance in [Be, section 3]. For example we can consider  $BSU \rightarrow BU$ ,  $BU \rightarrow BSO$  or  $B\mathrm{String} \rightarrow B\mathrm{Spin}$  to get  $\mathrm{THH}^{[n], MSU}(MU)$ ,  $\mathrm{THH}^{[n], MU}(MSO)$  or  $\mathrm{THH}^{[n], M\mathrm{String}}(M\mathrm{Spin})$ . As these examples give rise to Hopf-Galois extensions of ring spectra (see [R]) but not Galois extensions, the above relative  $\mathrm{THH}$ -terms will be non-trivial.

**4.2.  $\mathrm{THH}^{[n], HA}(\mathbb{F}_p)$  for commutative pointed  $\mathbb{F}_p$ -monoid algebras  $A$ .** Hesselholt and Madsen [HM, Theorem 7.1] showed a splitting result for topological Hochschild homology of pointed monoid rings. There is a straightforward generalization of this splitting result to higher order topological Hochschild homology in the commutative case. Let  $\Pi$  be a discrete pointed commutative monoid, *i.e.*, a commutative monoid in the category of based spaces with smash product. Assume moreover that  $\Pi$  is augmented, that is: admits a map of pointed monoids to the pointed monoid  $\{1, *\}$ , where  $1$  is the unit and  $*$  the base point. As long as  $1 \neq *$  in  $\Pi$ , there always is such a map: we can send all invertible elements in the monoid to  $1$  and all the rest to  $*$ . In general, however, such an augmentation is not unique, so it needs to be part of the data. We consider the monoid algebra where the ground ring is the field  $\mathbb{F}_p$  and all unadorned tensor products are understood to be over  $\mathbb{F}_p$ . Hesselholt-Madsen define  $\mathbb{F}_p[\Pi]$  as the  $\mathbb{F}_p$  vector space with basis  $\Pi$  modulo the subspace generated by the basepoint.

The analogue of [HM, Theorem 7.1] is a splitting of augmented commutative  $H\mathbb{F}_p$ -algebras:

$$(4.1.2) \quad \mathrm{THH}^{[n]}(H\mathbb{F}_p[\Pi]) \cong \mathrm{THH}^{[n]}(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathrm{THH}^{[n], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi]).$$

Note that  $\pi_*(\mathrm{THH}^{[n], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi])) \cong \mathrm{HH}_*^{\mathbb{F}_p, [n]}(\mathbb{F}_p[\Pi])$ .

**Theorem 4.2.** *For any  $\mathbb{F}_p$ -algebra  $\mathbb{F}_p[\Pi]$  on an augmented commutative pointed monoid  $\Pi$  with  $*$   $\neq 1$  there is a weak equivalence*

$$\mathrm{THH}^{[n], H\mathbb{F}_p[\Pi]}(H\mathbb{F}_p) \simeq H\mathbb{F}_p \wedge_{\mathrm{THH}^{[n], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi])}^L H\mathbb{F}_p[\Pi]$$

*of commutative augmented  $H\mathbb{F}_p$ -algebras.*

*Proof.* Theorem 3.2 applied to a model of the augmentation map  $H\mathbb{F}_p[\Pi] \rightarrow H\mathbb{F}_p$  that is a cofibration yields that

$$(4.2.1) \quad \mathrm{THH}^{[n], H\mathbb{F}_p[\Pi]}(H\mathbb{F}_p) \simeq \mathrm{THH}^{[n]}(H\mathbb{F}_p) \wedge_{\mathrm{THH}^{[n]}(H\mathbb{F}_p[\Pi])}^L H\mathbb{F}_p[\Pi].$$

We use the two-sided bar construction as model for the above derived smash product and use the splittings in (4.1.2) and (4.2.1) to obtain

$$\begin{aligned} & B(\mathrm{THH}^{[n]}(H\mathbb{F}_p), \mathrm{THH}^{[n]}(H\mathbb{F}_p[\Pi]), H\mathbb{F}_p[\Pi]) \\ & \simeq B(\mathrm{THH}^{[n]}(H\mathbb{F}_p), \mathrm{THH}^{[n]}(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathrm{THH}^{[n], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi]), H\mathbb{F}_p[\Pi]) \\ & \simeq B(\mathrm{THH}^{[n]}(H\mathbb{F}_p), \mathrm{THH}^{[n]}(H\mathbb{F}_p), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B(H\mathbb{F}_p, \mathrm{THH}^{[n], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi]), H\mathbb{F}_p[\Pi]) \\ & \simeq H\mathbb{F}_p \wedge_{H\mathbb{F}_p} B(H\mathbb{F}_p, \mathrm{THH}^{[n], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi]), H\mathbb{F}_p[\Pi]) \end{aligned}$$

which is a model of  $H\mathbb{F}_p \wedge_{\mathrm{THH}^{[n], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi])}^L H\mathbb{F}_p[\Pi]$ .  $\square$

We apply the above result to special cases of pointed commutative monoids, where we can identify the necessary ingredients for the above result.

**Proposition 4.3.**

- (a) *Consider the polynomial algebra  $\mathbb{F}_p[x]$  over  $\mathbb{F}_p$  (with  $|x| = 0$ , augmented by sending  $x \mapsto 0$ ), and let  $B'_1(x) = \mathbb{F}_p[x]$  and  $B'_{n+1}(x) = \mathrm{Tor}_{*,*}^{B_n^{(x)}}(\mathbb{F}_p, \mathbb{F}_p)$  with total grading. Then*

$$\mathrm{THH}_*^{[n], H\mathbb{F}_p[x]}(H\mathbb{F}_p) \cong B'_{n+2}(x).$$

- (b) *Let  $m$  be a natural number such that  $p$  divides  $m$ , and let  $B''_1(m) = \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$  with  $|\varepsilon x| = 1$  and  $|\varphi^0 x| = 2$  and  $B''_{n+1}(m) = \mathrm{Tor}^{B_n^{(m)}}(\mathbb{F}_p, \mathbb{F}_p)$ . Then*

$$\mathrm{THH}_*^{[n], H\mathbb{F}_p[x]/x^m}(H\mathbb{F}_p) \cong B''_{n+1}(m).$$

- (c) *Let  $G$  be a finitely generated abelian group, so,  $G = \mathbb{Z}^m \oplus \bigoplus_{i=1}^N \mathbb{Z}/q_i^{\ell_i}$  for some primes  $q_i$ . Then  $\mathrm{THH}_*^{[n], H\mathbb{F}_p[G]}(H\mathbb{F}_p)$  can be expressed in terms of a tensor product of factors that are isomorphic to  $\mathrm{THH}_*^{[n], H\mathbb{F}_p[x]}(H\mathbb{F}_p)$  or  $\mathrm{THH}_*^{[n], H\mathbb{F}_p[x]/(x^{p^\ell})}(H\mathbb{F}_p)$  for some  $\ell$ .*

*Proof.* We can rewrite  $H\mathbb{F}_p \wedge_{\mathrm{THH}^{[n], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi])}^L H\mathbb{F}_p[\Pi]$  as

$$H\mathbb{F}_p \wedge_{H\mathbb{F}_p[\Pi]} H\mathbb{F}_p[\Pi] \wedge_{\mathrm{THH}^{[n], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi])}^L H\mathbb{F}_p[\Pi]$$

which is equivalent to

$$H\mathbb{F}_p \wedge_{H\mathbb{F}_p[\Pi]} \mathrm{THH}^{[n+1], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi]).$$

In [BLPRZ]  $\pi_*(\mathrm{THH}^{[n+1], H\mathbb{F}_p}(H\mathbb{F}_p[\Pi])) \cong \mathrm{HH}_*^{[n+1], \mathbb{F}_p}(\mathbb{F}_p[\Pi])$  is calculated in the cases of the Proposition:

For (a) we consider the pointed monoid  $\Pi = \{0, 1, x, x^2, \dots\}$  whose associated pointed monoid ring  $\mathbb{F}_p[\Pi]$  is the ring of polynomials over  $\mathbb{F}_p$ . In [BLPRZ, Theorem 8.6], we show inductively that

$$\mathrm{HH}_*^{[n], \mathbb{F}_p}(\mathbb{F}_p[x]) \cong \mathbb{F}_p[x] \otimes B'_{n+1}(x).$$

We also get inductively that the augmentation on  $\mathrm{HH}_*^{[n]}(\mathbb{F}_p[x])$  is the identity on the  $\mathbb{F}_p[x]$  factor and for degree reasons the obvious augmentation on  $B'_{n+1}(x)$ . Therefore the claim follows.

Higher Hochschild homology of truncated polynomial algebras of the form  $\mathbb{F}_p[x]/x^m$  for  $m$  divisible by  $p$  was calculated in [BLPRZ] (the case  $m = p^\ell$ ) and [BHLPRZ] (the general case). The result in those cases is

$$\mathrm{HH}_*^{[n]}(H\mathbb{F}_p[x]/x^m) \cong \mathbb{F}_p[x]/x^m \otimes B_n''(m)$$

where  $B_1''(m) = \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$  with  $|\varepsilon x| = 1$  and  $|\varphi^0 x| = 2$  and where  $B_{n+1}''(m) = \mathrm{Tor}^{B_n''(m)}(\mathbb{F}_p, \mathbb{F}_p)$ . This implies (b).

For a finitely generated abelian group as in (c) the group ring splits as

$$\mathbb{F}_p[\mathbb{Z}]^{\otimes m} \otimes \bigotimes_{i=1}^N \mathbb{F}_p[\mathbb{Z}/q_i^{\ell_i}].$$

The torsion groups with torsion prime to  $p$  do not contribute to higher (topological) Hochschild homology because they are étale over  $\mathbb{F}_p$  (see [BLPRZ, Theorem 9.1] and [H, Theorem 7.9]). For the free factors we use the fact that  $\mathbb{F}_p[\mathbb{Z}] = \mathbb{F}_p[x^{\pm 1}]$  is étale over  $\mathbb{F}_p[x]$ ; for the factors with torsion that is a power of  $p$ , we use the fact that  $\mathbb{F}_p[\mathbb{Z}/p^\ell] \cong \mathbb{F}_p[y]/(y^{p^\ell} - 1) \cong \mathbb{F}_p[x]/(x^{p^\ell})$  by taking  $x = y - 1$ .  $\square$

*Remark 4.4.* Let  $k$  be a commutative ring and let  $A$  be a commutative  $k$ -algebra. In [BHLPRZ] we define higher order Shukla homology of  $A$  over  $k$  as

$$\mathrm{Shukla}_*^{[n],k}(A) := \mathrm{THH}_*^{[n],Hk}(HA).$$

Thus the calculations above determine higher order Shukla homology for commutative pointed monoid algebras over  $\mathbb{F}_p$ ,  $\mathrm{Shukla}_*^{[n],\mathbb{F}_p[\Pi]}(\mathbb{F}_p)$ .

**4.3. The examples  $ko$ ,  $ku$ ,  $\ell$  and  $tmf$ .** Angeltveit and Rognes calculate in [AR, 5.13, 6.2]  $H_*(\mathrm{THH}(ko); \mathbb{F}_2)$ ,  $H_*(\mathrm{THH}(tmf); \mathbb{F}_2)$ ,  $H_*(\mathrm{THH}(ku); \mathbb{F}_2)$  and for any odd prime  $p$  they determine  $H_*(\mathrm{THH}(\ell); \mathbb{F}_p)$  where  $\ell \rightarrow ku_{(p)}$  is the Adams summand of  $p$ -local connective topological complex K-theory.

The following lemma collects some immediate consequences of their work, which were already noticed in [G]. These will be the basis of the calculations in the results that follow the lemma. The index of a generator denotes its degree.

**Lemma 4.5.**

(a)

$$\mathrm{THH}_*(ko; \mathbb{F}_2) \cong \Lambda(x_5, x_7) \otimes \mathbb{F}_2[\mu_8].$$

(b)

$$\mathrm{THH}_*(tmf; \mathbb{F}_2) \cong \Lambda(x_9, x_{13}, x_{15}) \otimes \mathbb{F}_2[\mu_{16}].$$

(c)

$$\mathrm{THH}_*(ku; \mathbb{F}_2) \cong \Lambda(x_3, x_7) \otimes \mathbb{F}_2[\mu_8].$$

(d) *At any odd prime:*

$$\mathrm{THH}_*(\ell; \mathbb{F}_p) \cong \Lambda(x_{2p-1}, x_{2p^2-1}) \otimes \mathbb{F}_p[y_{2p^2}].$$

*Proof.* In all four cases Angeltveit and Rognes show that  $H_*(\mathrm{THH}(E); \mathbb{F}_p)$  is of the form  $H_*(E; \mathbb{F}_p) \otimes A_E$  for  $A_E$  as described in (4.5.1) below, where  $p = 2$  for  $E = ko, tmf, ku$  and  $p$  is odd for  $E = \ell$ . We rewrite  $\pi_*(\mathrm{THH}(E; H\mathbb{F}_p))$  as

$$\begin{aligned} \pi_*(\mathrm{THH}(E; H\mathbb{F}_p)) &\cong \pi_*(\mathrm{THH}(E) \wedge_E^L H\mathbb{F}_p) \\ &\cong \pi_*((\mathrm{THH}(E) \wedge H\mathbb{F}_p) \wedge_{E \wedge H\mathbb{F}_p}^L H\mathbb{F}_p) \end{aligned}$$

and thus we get a spectral sequence

$$E_{s,t}^2 = \mathrm{Tor}_{s,t}^{H_*(E; \mathbb{F}_p)}(H_*(\mathrm{THH}(E); \mathbb{F}_p), \mathbb{F}_p)$$

converging to the homotopy groups of  $\mathrm{THH}(E; H\mathbb{F}_p)$ . As  $H_*(\mathrm{THH}(E); \mathbb{F}_p) \cong H_*(E; \mathbb{F}_p) \otimes A_E$  in all four cases, the  $E^2$ -term above is concentrated in the  $s = 0$  column with

$$E_{0,*}^2 \cong A_E,$$

where

$$(4.5.1) \quad A_E = \begin{cases} \Lambda_{\mathbb{F}_2}(\sigma \bar{\xi}_1^4, \sigma \bar{\xi}_2^2) \otimes \mathbb{F}_2[\sigma \bar{\xi}_3], & E = ko, \\ \Lambda_{\mathbb{F}_2}(\sigma \bar{\xi}_1^8, \sigma \bar{\xi}_2^4, \sigma \bar{\xi}_3^2) \otimes \mathbb{F}_2[\sigma \bar{\xi}_4], & E = tmf, \\ \Lambda_{\mathbb{F}_2}(\sigma \bar{\xi}_1^2, \sigma \bar{\xi}_2^2) \otimes \mathbb{F}_2[\sigma \bar{\xi}_3], & E = ku, \text{ and} \\ \Lambda_{\mathbb{F}_p}(\sigma \bar{\xi}_1, \sigma \bar{\xi}_2) \otimes \mathbb{F}_p[\sigma \bar{\tau}_2], & E = \ell. \end{cases}$$

The degrees are the usual degrees in the dual of the Steenrod algebra, hence at 2 we have  $|\xi_i| = 2^i - 1$  and at odd primes  $|\xi_i| = 2p^i - 2$  and  $|\tau_i| = 2p^i - 1$ . We also have  $|\sigma y| = |y| + 1$ . The  $(\bar{\cdot})$  denotes conjugation in the dual of the Steenrod algebra. Counting degrees gives the claim.  $\square$

We can use the equivalence  $\mathrm{THH}^A(B) \simeq B \wedge_{\mathrm{THH}(A;B)}^L \mathrm{THH}(B)$  from Theorem 3.2 to deduce the following result.

**Theorem 4.6.** *There are additive isomorphisms*

- (a)  $\mathrm{THH}^{ko}(H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_5) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \mathbb{F}_2[\mu_2]/\mu_2^4,$
- (b)  $\mathrm{THH}^{tmf}(H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_9) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_{13}) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_{15}) \otimes \mathbb{F}_2[\mu_2]/\mu_2^8,$
- (c)  $\mathrm{THH}^{ku}(H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_3) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \mathbb{F}_2[\mu_2]/\mu_2^4,$
- (d) *and for odd primes  $p$  we get an additive isomorphism*

$$\mathrm{THH}^\ell(H\mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(\rho^0 x_{2p-1}) \otimes \Gamma_{\mathbb{F}_p}(\rho^0 x_{2p^2-1}) \otimes \mathbb{F}_p[\mu_2]/\mu_2^{p^2}.$$

Here  $\rho^0$  raises degree by one.

*Proof.* We use Theorem 3.2 in the case where  $B = H\mathbb{F}_p$ . In [B], Bökstedt shows that  $\mathrm{THH}_*(H\mathbb{F}_p) \cong \mathbb{F}_p[\mu_2]$  for all primes  $p$ . We give the details for the case  $ko$ ; the arguments for the other examples are completely analogous.

The  $E^2$ -term of the spectral sequence is

$$\mathrm{Tor}_*^{\Lambda(x_5, x_7) \otimes \mathbb{F}_2[\mu_8]}(\mathbb{F}_2, \mathbb{F}_2[\mu_2]) \implies \mathrm{THH}_*^{ko}(H\mathbb{F}_2).$$

Since both  $x_5$  and  $x_7$  have odd degrees, they cannot act on  $\mu_2$  other than trivially. Thus we can rewrite the left-hand side as

$$\mathrm{Tor}_*^{\Lambda(x_5, x_7)}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Tor}_*^{\mathbb{F}_2[\mu_8]}(\mathbb{F}_2, \mathbb{F}_2[\mu_2]).$$

The explicit description of the generators in [AR, Theorem 6.2] implies that the map  $\mathbb{F}_2[\mu_8] \rightarrow \mathbb{F}_2[\mu_2]$  takes  $\mu_8$  to  $\mu_2^4$ , because  $\mu_8$  corresponds to  $\sigma \bar{\xi}_3$  and Angeltveit and Rognes show [AR, proof of 5.12] that the  $\sigma \bar{\xi}_k$  satisfy

$$(\sigma \bar{\xi}_k)^2 = \sigma \bar{\xi}_{k+1}$$

for  $p = 2$  and  $\mu_2$  in Bökstedt's calculation corresponds to  $\sigma \bar{\xi}_1$ .

Therefore the right-hand  $\mathrm{Tor}$  is isomorphic to  $\mathbb{F}_2[\mu_2]/\mu_2^4$ . Hence the  $E^2$ -term is isomorphic to

$$\Gamma_{\mathbb{F}_2}(\rho^0 x_5) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \mathbb{F}_2[\mu_2]/\mu_2^4.$$

Since all the nonzero classes in this  $E^2$ -term have even total degree, the spectral sequence must collapse at  $E^2$ .

In the case of the Adams summand,  $\ell$ , we work at odd primes and here in [AR, proof of 5.12] the relation

$$(\sigma \bar{\tau}_k)^p = \sigma \bar{\tau}_{k+1}$$

is shown. Hence  $\sigma \bar{\tau}_2$  in  $\mathrm{THH}_*(\ell; H\mathbb{F}_p)$  corresponds to  $(\sigma \bar{\tau}_0)^{p^2}$  and  $\sigma \bar{\tau}_0$  is the element that represents  $\mu_2$  at odd primes.  $\square$

*Remark 4.7.* In order to determine for instance  $\mathrm{THH}^{ko}(H\mathbb{F}_2)$  multiplicatively we would also need to control possible multiplicative extensions. We can show by degree considerations that  $(\rho^k x_7)^2 = 0$ , but not whether  $(\rho^k x_5)^2$  must vanish.

As a general warning we discuss the case of the bar spectral sequence in the case  $\mathrm{THH}_*(ku; H\mathbb{F}_2)$ . Here, we know the answer from Lemma 4.5:

$$\mathrm{THH}_*(ku; H\mathbb{F}_2) \cong \Lambda(x_3, x_7) \otimes \mathbb{F}_2[\mu_8].$$

However, if we use the bar spectral sequence we get as the  $E^2$ -term

$$\mathrm{Tor}_{*,*}^{\mathrm{THH}_*^{[0]}(ku; H\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Tor}_{*,*}^{\mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_k, k \geq 3]}(\mathbb{F}_2, \mathbb{F}_2)$$

because

$$\mathrm{THH}_*^{[0]}(ku; H\mathbb{F}_2) \cong H_*(ku; \mathbb{F}_2) \cong \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_k, k \geq 3]$$

(see for instance [AR, 6.1]). Hence the spectral sequence collapses because

$$E_{*,*}^2 \cong \Lambda_{\mathbb{F}_2}(\sigma \bar{\xi}_1^2, \sigma \bar{\xi}_2^2, \sigma \bar{\xi}_k, k \geq 3)$$

and all generators are concentrated on the 1-line. But we know that the exterior generators in  $\Lambda_{\mathbb{F}_2}(\sigma \bar{\xi}_k, k \geq 3)$  extend to form  $\mathbb{F}_2[\mu_8]$ , so there are highly non-trivial multiplicative extensions in this spectral sequence.

*Remark 4.8.* Veen established a Hopf-structure on the bar spectral sequence [V, §7] for higher order  $\mathrm{THH}$  of  $H\mathbb{F}_p$ . His argument generalizes: The pinch maps  $\mathbb{S}^n \rightarrow \mathbb{S}^n \vee \mathbb{S}^n$  give rise to a comultiplication

$$\mathrm{THH}^{[n]}(A; C) \rightarrow \mathrm{THH}^{[n]}(A; C) \wedge_C \mathrm{THH}^{[n]}(A; C)$$

and as the multiplication on  $\mathrm{THH}^{[n]}(A; C)$  is induced by the fold map, both structures are compatible. For  $\mathrm{THH}(A)$  this structure is heavily used in [AR].

If  $A$  is connective, then we can consider  $A \rightarrow H(\pi_0 A)$ . For  $C = H\mathbb{F}_p$  this multiplication and comultiplication turns  $\mathrm{THH}_*^{[n]}(A; H\mathbb{F}_p)$  into an  $\mathbb{F}_p$ -Hopf algebra. Veen's arguments also transfer to yield that the bar spectral sequence

$$\mathrm{Tor}_{*,*}^{\mathrm{THH}_*^{[n]}(A; H\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \mathrm{THH}_*^{[n+1]}(A; H\mathbb{F}_p)$$

is a spectral sequence of Hopf-algebras; in particular, the differentials satisfy a Leibniz and a co-Leibniz rule and these facts let us determine the differentials in certain cases.

**Theorem 4.9.** *Additively,*

$$\mathrm{THH}^{[2]}(ko; H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_5) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \Lambda_{\mathbb{F}_2}(\epsilon \mu_8).$$

Here, the degrees are  $|\rho^0 x_5| = 6$ ,  $|\rho^0 x_7| = 8$  and  $|\epsilon \mu_8| = 9$ .

*Proof.* Using the Tor spectral sequence we get

$$\mathrm{Tor}_{*,*}^{\Lambda(x_5) \otimes \Lambda(x_7) \otimes \mathbb{F}_2[\mu_8]}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \mathrm{THH}_*^{[2]}(ko, H\mathbb{F}_2).$$

The  $E^2$  page of the spectral sequence is of the form  $\Gamma(\rho^0 x_5) \otimes \Gamma(\rho^0 x_7) \otimes \Lambda(\epsilon \mu_8)$ . This, in turn, is isomorphic to

$$\bigotimes_{k=0}^{\infty} \underbrace{\mathbb{F}_2(\rho^k x_5) / (\rho^k x_5)^2}_{\Lambda(\rho^k x_5)} \otimes \bigotimes_{\ell=0}^{\infty} \underbrace{\mathbb{F}_2(\rho^\ell x_7) / (\rho^\ell x_7)^2}_{\Lambda(\rho^\ell x_7)} \otimes \Lambda(\epsilon \mu_8),$$

with bidegrees  $\|\epsilon \mu_8\| = (1, 8)$ , and  $\|\rho^k x_i\| = (2^k, 2^k i)$ . The claim is that the spectral sequence collapses at  $E^2$ .

As the spectral sequence above is a spectral sequence of Hopf algebras, the smallest nonzero differential must go from an indecomposable element to a primitive element. As the only primitive element in  $\Gamma(\rho^0 x_i)$  is  $\rho^0 x_i$  we just need to check that no differentials hit  $\rho^0 x_i$ ,  $i = 5, 7$ , or  $\epsilon \mu_8$ . These have bidegrees  $(1, i)$  and  $(1, 8)$ , respectively, and thus if they are hit by  $d^r$  of an indecomposable  $d^r(\rho^k x_j)$ , which would have bidegree  $(2^k - r, 2^k j + r - 1)$ , we must have



$2^k - r = 1$  and  $2^k j + r - 1 = 5, 7$ , or  $8$ . Substituting  $r = 2^k - 1$  into the second expression, it becomes  $2^k(j + 1) - 2$ , where clearly  $k \geq 1$  and  $j = 5$  or  $7$ , making  $2^k(j + 1) - 2 \geq 10$ . So the spectral sequence collapses at  $E^2$ .  $\square$

So far, we cannot rule out non-trivial multiplicative extensions. These could be nontrivial only if there is a multiplicative generator  $\rho^k x_i$  or  $\epsilon \mu_8$  with whose square is not zero in  $\mathrm{THH}_*^{[2]}(ko, H\mathbb{F}_2)$  (although it is zero in the  $E^\infty$  term). This is possible only if  $(\rho^k x_i)^2$  has filtration degree less than  $2^{k+1}$ , or  $(\epsilon \mu_8)^2$  has filtration degree less than  $2$ . The latter is clearly impossible, since there is nothing in total degree  $18$  in filtration degrees  $0$  or  $1$ . If  $i = 7$ , by similar arguments we cannot have anything in total degree  $2^{k+1} \cdot 8$  in filtration degree less than  $2^{k+1}$ : it would have to be constructed out of elements in bidegrees  $(a, 5a)$ ,  $(b, 7b)$  and possibly also one occurrence of the element of bidegree  $(1, 8)$ , but then the first coordinates would have to add up to at least  $2^{k+1}$  to have the total degree equal to  $2^{k+1} \cdot 8$ . However, for the cases  $(\rho^k x_5)^2$  one cannot rule out extensions just for degree reasons.

In a similar manner as above, we can exclude non-trivial differentials in the other three cases:

**Proposition 4.10.** *There are additive isomorphisms*

- (a)  $\mathrm{THH}_*^{[2]}(ku; H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_3) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \Lambda_{\mathbb{F}_2}(\epsilon \mu_8)$ ,
- (b)  $\mathrm{THH}_*^{[2]}(tmf; H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_9) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_{13}) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_{15}) \otimes \Lambda_{\mathbb{F}_2}(\epsilon \mu_{16})$ ,
- (c) and for any odd prime  $p$  we get an additive isomorphism

$$\mathrm{THH}_*^{[2]}(\ell; H\mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(\rho^0 x_{2p-1}) \otimes \Gamma_{\mathbb{F}_p}(\rho^0 x_{2p^2-1}) \otimes \Lambda_{\mathbb{F}_p}(\epsilon \mu_{2p^2}).$$

*Proof.*

- (a) In the case of  $ku$  at the even prime we get a degree constraint for a differential  $d^r(\rho^k x_i)$  of the form

$$(2^k - r, 2^k i + r - 1) = (1, j)$$

where  $j$  is  $3, 7$  or  $8$ . Since  $r \geq 2$  and  $2^k - r = 1$ , we get  $k \geq 2$ , but that would make the internal degree at least  $4(i + 1) - 2$  and this is bigger or equal to  $14$ , hence doesn't occur.

- (b) For  $tmf$  the degree constraint is

$$(2^k - r, 2^k i + r - 1) = (1, j)$$

where  $j$  is  $9, 13, 15$  or  $16$ . Again  $r \geq 2$  and  $2^k - r = 1$  imply that  $k \geq 2$ , which makes the internal degree at least  $38$ .

- (c) For the Adams summand  $\ell$  the degree condition is

$$(p^k - r, p^k i + r - 1) = (1, j)$$

where  $j = 2p - 1, 2p^2 - 1$  or  $2p^2$ . As before, we get that  $k \geq 2$  and therefore we get an internal degree of at least  $2p^3 - 2$  which is too big to be the degree of a primitive element.

In all cases the differentials  $d^r$ ,  $r \geq 2$ , all have to be trivial and we get the result.  $\square$

**4.4. A splitting for  $\mathrm{THH}^{[n], Hk}(HA)$  for commutative  $k$ -algebras  $A$ .** We apply Theorem 3.6 for a sequence of cofibrations of commutative  $S$ -algebras of the form  $S \rightarrow A \rightarrow B = C$  and as  $\mathrm{THH}^{[n-1], B}(B) \simeq B$  we obtain a weak equivalence

$$(4.10.1) \quad \mathrm{THH}^{[n]}(B) \simeq \mathrm{THH}^{[n]}(A; B) \wedge_{\mathrm{THH}^{[n-1], A}(B)}^L B.$$

In the special case of a sequence  $S \rightarrow Hk \rightarrow HA = HA$  where  $A$  is a commutative  $k$ -algebra the formula in (4.10.1) specializes to the following result.

**Proposition 4.11.** *For all commutative rings  $k$  and all commutative  $k$ -algebras  $A$  the higher topological Hochschild homology of  $HA$  splits as*

$$\mathrm{THH}^{[n]}(HA) \simeq \mathrm{THH}^{[n]}(Hk; HA) \wedge_{\mathrm{Shukla}^{[n-1],k}(A)}^L HA.$$

*If  $A$  is flat as a  $k$ -module, then higher Shukla homology reduces to higher Hochschild homology and we obtain*

$$\mathrm{THH}^{[n]}(HA) \simeq \mathrm{THH}^{[n]}(Hk; HA) \wedge_{\mathrm{HH}^{[n-1],k}(A)}^L HA.$$

In particular, this gives splitting results for number rings: For  $k = \mathbb{Z}$  and  $A = \mathcal{O}_K$  a ring of integers in a number field we get

$$\mathrm{THH}^{[n]}(H\mathcal{O}_K) \simeq \mathrm{THH}^{[n]}(H\mathbb{Z}; H\mathcal{O}_K) \wedge_{\mathrm{HH}^{[n-1],\mathbb{Z}(\mathcal{O}_K)}}^L H\mathcal{O}_K.$$

The (topological) Hochschild homology of  $\mathcal{O}_K$  is known (see [LL, LM]). However, the additive and multiplicative structure of these is complicated enough that we cannot use the iteration methods of [BLPRZ, DLR] and so we do not know the higher order topological Hochschild homology of  $\mathcal{O}_K$  with unreduced coefficients so far, nor its higher Shukla homology.

*Remark 4.12.* Beware that the splitting

$$\begin{aligned} \mathrm{THH}^{[n]}(HA) &\simeq \mathrm{THH}^{[n]}(Hk; HA) \wedge_{\mathrm{Shukla}^{[n-1],k}(A)}^L HA \\ &\simeq (\mathrm{THH}^{[n]}(Hk) \wedge_{Hk} HA) \wedge_{\mathrm{Shukla}^{[n-1],k}(A)}^L HA \end{aligned}$$

cannot be rearranged to

$$\mathrm{THH}^{[n]}(Hk) \wedge_{Hk} (HA \wedge_{\mathrm{Shukla}^{[n-1],k}(A)}^L HA) = \mathrm{THH}^{[n]}(Hk) \wedge_{Hk} \mathrm{Shukla}^{[n],k}(A)$$

because the  $\mathrm{Shukla}^{[n-1],k}(A)$ -action on  $\mathrm{THH}^{[n]}(Hk; HA)$  does not usually factor through an action on the coefficients  $HA$ . If we could rearrange it that way, it would imply that  $\mathrm{Shukla}^{[n],k}(A)$  splits off  $\mathrm{THH}^{[n]}(HA)$ , which is not true even for  $n = 1$ : for example, if we take  $k = \mathbb{Z}$  and  $A = \mathbb{Z}[i]$ , since  $\mathrm{THH}(H\mathbb{Z})$  as the topological Hochschild homology of a ring is equivalent to a product of Eilenberg Mac Lane spectra, which Bökstedt [B] identified to be

$$\mathrm{THH}(H\mathbb{Z}) \simeq H\mathbb{Z} \times \prod_{a=2}^{\infty} \Sigma^{2a-1} H(\mathbb{Z}/a\mathbb{Z}),$$

then we get the formula

$$\begin{aligned} \pi_*(\mathrm{THH}(H\mathbb{Z}) \wedge_{H\mathbb{Z}} \mathrm{Shukla}(\mathbb{Z}[i])) \\ &\cong \pi_*(\mathrm{Shukla}(\mathbb{Z}[i])) \oplus \bigoplus_{a=2}^{\infty} \pi_*(\Sigma^{2a-1} H(\mathbb{Z}/a\mathbb{Z}) \wedge_{H\mathbb{Z}} \mathrm{Shukla}(\mathbb{Z}[i])) \\ &\cong \mathrm{HH}_*(\mathbb{Z}[i]) \oplus \bigoplus_{a=2}^{\infty} \left( \mathrm{HH}_{*-2a+1}(\mathbb{Z}[i]) \otimes \mathbb{Z}/a\mathbb{Z} \oplus \mathrm{Tor}(\mathrm{HH}_{*-2a}(\mathbb{Z}[i]), \mathbb{Z}/a\mathbb{Z}) \right) \end{aligned}$$

where  $\mathrm{HH}_*(\mathbb{Z}[i]) = 0$  when  $*$  < 0. We also know that  $\mathrm{HH}_0(\mathbb{Z}[i]) \cong \mathbb{Z}[i]$ ,  $\mathrm{HH}_{2a-1}(\mathbb{Z}[i]) \cong \mathbb{Z}[i]/2\mathbb{Z}[i]$ , and the positive even groups vanish. Thus the number of copies of  $\mathbb{Z}/2\mathbb{Z}$ 's in  $\pi_n(\mathrm{THH}(H\mathbb{Z}) \wedge_{H\mathbb{Z}} \mathrm{Shukla}(\mathbb{Z}[i]))$  grows linearly with  $n$ . On the other hand, by [L],  $\mathrm{THH}_0(\mathbb{Z}[i]) \cong \mathbb{Z}[i]$ ,  $\mathrm{THH}_{2a-1}(\mathbb{Z}[i]) \cong \mathbb{Z}[i]/2a\mathbb{Z}[i]$ , and the positive even groups vanish.

Such a splitting of  $\mathrm{Shukla}^{[n],k}(A)$  off  $\mathrm{THH}^{[n]}(HA)$  does hold under additional assumptions, for instance in the case of commutative pointed monoid rings (see (4.1.2) above).

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SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SHEFFIELD, HICKS BUILDING, HOUNSFIELD ROAD, S3 7RH SHEFFIELD, UK

*E-mail address:* glhalliwell1@sheffield.ac.uk

INSTITUT GALILÉE, UNIVERSITÉ PARIS 13, 99, AVENUE JEAN-BAPTISTE CLÉMENT, 93430 VILLETANEUSE, FRANCE

*E-mail address:* hoening@math.univ-paris13.fr

MATHEMATICS DEPARTMENT, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405, USA

*E-mail address:* alindens@indiana.edu

FACHBEREICH MATHEMATIK DER UNIVERSITÄT HAMBURG, BUNDESSTRASSE 55, 20146 HAMBURG, GERMANY

*E-mail address:* birgit.richter@uni-hamburg.de

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CHICAGO, 5734 S UNIVERSITY AVE, CHICAGO, IL 60637, USA

*E-mail address:* zakh@math.uchicago.edu